Løsn. innlev 2 foa130 h2004 - maskin/marin

Nr 1 a)

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = \int_0^1 f(t)e^{-st}dt + \int_1^\infty f(t)e^{-st}dt =$$

$$= \int_0^1 e^t \cdot e^{-st}dt + \int_1^\infty e^{-t+1} \cdot e^{-st}dt$$

$$= \int_0^1 e^{-t(s-1)}dt + \int_1^\infty e \cdot e^{-t(s+1)}dt$$

$$= \left[-\frac{1}{s-1}e^{-t(s-1)}\right]_0^1 + e\cdot\left[-\frac{1}{s+1}e^{-t(s+1)}\right]_1^\infty$$

$$= -\frac{1}{s-1}\cdot e^{-(s-1)} + \frac{1}{s-1} + e\left\{0 - \frac{-1}{s+1}e^{-(s+1)}\right\}$$

$$= \underline{\underline{\frac{1}{s-1} - \frac{e}{s-1}e^{-s} + \frac{1}{s+1}e^{-s}}} \quad *$$

b) (i) $u(t)(t-2)^2 = u(t)(t^2-4t+4) = u(t)t^2 - 4tu(t) + 4u(t)$

$$\Leftrightarrow \underline{\underline{\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}}} \quad \text{(Lineæritet og tabell)}$$

(ii) $u(t)\sin 2t \Leftrightarrow \underline{\frac{2}{s^2+4}}$ (direkte av tabell)

$$t\cdot u(t)\sin 2t \Leftrightarrow -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = (-1)2(-1)(s^2+4)^{-2}2s = \underline{\underline{\frac{4s}{(s^2+4)^2}}}$$

(iii) $e^{-2t} \cdot u(t)\cos t \Longleftrightarrow ?$

$$f(t) = \boxed{u(t)\cos t \Longleftrightarrow \frac{s}{s^2+1}} = F(s)$$

$\left\{\begin{array}{l} \text{å gange med } e^{-2t} \text{ bevirker forskyvning i } s \\ e^{at} f(t) \Longleftrightarrow F(s-a) \quad \text{altså med } a = -2 \end{array}\right.$

$$e^{-2t} \cdot u(t)\cos t \Longleftrightarrow \underline{\underline{\frac{s+2}{(s+2)^2+1}}}$$

(iv) $u(t-2)\,t^2 \Longleftrightarrow ?$ $\boxed{\text{forskyvning i tid p.g.a. } u(t-2)}$

$\left\{\begin{array}{l} t \to t+2 \quad u(t)(t+2)^2 = u(t)(t^2+4t+4) \Longleftrightarrow \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \\ \text{så ganges dette med } e^{-2s} \end{array}\right.$

$\therefore \quad u(t-2)\,t^2 \Longleftrightarrow \underline{\underline{\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) \cdot e^{-2s}}}$

(v) $u(t-1)\cos(t-1) \Longleftrightarrow ?$ $\boxed{\text{forskyvn. i tid pga. } u(t-1)}$

$t \to t+1$

$u(t)\cos t \Longleftrightarrow \frac{s}{s^2+1}$, så ganges dette med e^{-s}

$\therefore \quad u(t-1)\cos(t-1) \Longleftrightarrow \underline{\underline{\frac{s}{s^2+1} \cdot e^{-s}}}$

Nr2 (i) $\frac{1}{(s-2)^2} \Leftrightarrow ?$ $\boxed{\text{forskyvning i } s}$

$s \to s+2$

$\frac{1}{s^2} \leftrightarrow t\,u(t)$, så ganges dette med e^{2t}

$\therefore \frac{1}{(s-2)^2} \Leftrightarrow \underline{\underline{e^{2t} \cdot t\,u(t)}}$

(ii) $\frac{s}{(s-2)^2} \Leftrightarrow ?$ $\boxed{\text{forskyvning i } s}$

$s \to s+2$

$\frac{s+2}{s^2} = \frac{1}{s} + \frac{2}{s^2} \Leftrightarrow u(t) + 2t\,u(t)$, så gang med e^{2t}

$\therefore \frac{s}{(s-2)^2} \Leftrightarrow \underline{\underline{e^{2t} \cdot \{u(t) + 2t\,u(t)\}}}$

(iii) $\frac{1}{s}\left(e^{-2s}+1\right) = \frac{1}{s}e^{-2s} + \frac{1}{s}$

$\underline{\frac{1}{s} \Leftrightarrow u(t)}$. $\therefore \frac{1}{s}e^{-2s} \Leftrightarrow u(t-2)$

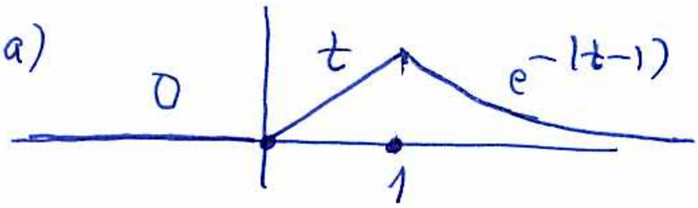
å gange med e^{-2s} bevirker forskyvning i t

Altså $\frac{1}{s}\left(e^{-2s}+1\right) \Leftrightarrow \underline{\underline{u(t-2)+u(t)}}$

Nr 3

$$f(t) = \begin{cases} 0 & t<0 \\ t & 0<t<1 \qquad u(t)-u(t-1) \\ e^{-(t-1)} & t>1 \qquad u(t-1) \end{cases}$$

a)



0 og 1 er to kritiske argument.

$$\underline{\underline{f(t) = t\cdot\left[u(t)-u(t-1)\right] + e^{-(t-1)}\cdot u(t-1)}}$$

b) $f(t) = t\,u(t) - t\,u(t-1) + e^{-(t-1)}\cdot u(t-1)$

$t\,u(t) \updownarrow \frac{1}{s^2}$, $\quad t\,u(t-1) \updownarrow ?$, $\quad e^{-(t-1)}\cdot u(t-1) \updownarrow ?$

undersøker disse nærmere

$$\left\{ \begin{array}{l} t\,u(t-1) \Leftrightarrow \left(\frac{1}{s^2}+\frac{1}{s}\right)e^{-s} \\ t \to t+1 \\ (t+1)\,u(t) \Leftrightarrow \frac{1}{s^2}+\frac{1}{s} \end{array} \right.$$

$$\begin{cases} e^{-(t-1)}\,u(t-1) \Leftrightarrow \dfrac{1}{s+1}\cdot e^{-s} \\ t \to t+1 \\ e^{-t}\cdot u(t) \Leftrightarrow \dfrac{1}{s+1} \end{cases}$$

Da får vi $\mathcal{L}[f(t)] = \dfrac{1}{s^2} - \left(\dfrac{1}{s^2}+\dfrac{1}{s}\right)e^{-s} + \dfrac{1}{s+1}\cdot e^{-s}$

c) (i) $\dfrac{1}{s(s+1)} = \dfrac{1}{s} - \dfrac{1}{s+1} \Leftrightarrow u(t) - u(t)\,e^{-t}$

(ii) $\dfrac{1}{s(s+1)}\cdot e^{-2s} \Leftrightarrow u(t-2) - u(t-2)\,e^{-(t-2)}$

bevirker at $t \to t-2$, se (i)

(iii) $\dfrac{1}{s^2+2}\cdot\dfrac{1}{s+1} = \dfrac{As+B}{s^2+2} + \dfrac{C}{s+1} = \dfrac{(As+B)(s+1)+C(s^2+2)}{(s^2+2)(s+1)}$

teller = teller ; nevner = nevner, altså:

$$1 = (A+C)s^2 + (A+B)s + 2C + B$$

s^2: $A+C=0$ $\quad C=-A$ inn i siste ligning.

$$\begin{rcases} s:\ A+B=0 \\ 1:\ 2C+B=1 \end{rcases} \quad \begin{array}{l} A+B=0 \\ -2A+B=1 \end{array} \Big|\, 2 \qquad 3B=1 \quad B=1/3$$

så $A = -B = -1/3$ så $C = -A = 1/3$

$\therefore$

$$\frac{1}{s^2+2}\cdot\frac{1}{s+1} = -\frac{1}{3}\,\frac{s}{s^2+2} + \frac{1}{3}\,\frac{1}{s^2+2} + \frac{1}{3}\,\frac{1}{s+1}$$

$$\longleftrightarrow \underline{\underline{\left[-\frac{1}{3}\cos\sqrt{2}t + \frac{1}{3}\frac{1}{\sqrt{2}}\sin\sqrt{2}t + \frac{1}{3}e^{-t}\right]u(t)}}$$

(iv) $\frac{1}{s^2+2}\cdot\frac{1}{s+1}\,e^{-s} \longleftrightarrow ?$

Vi fant $\frac{1}{s^2+2}\cdot\frac{1}{s+1} \longleftrightarrow g(t)$

med $g(t) = u(t)\cdot\left[-\frac{1}{3}\cos\sqrt{2}t + \frac{1}{3\sqrt{2}}\sin\sqrt{2}t + \frac{1}{3}e^{-t}\right]$

ovenfor.

Men da blir jo $\frac{1}{s^2+2}\cdot\frac{1}{s+1}\,e^{-s} \longleftrightarrow \underline{\underline{g(t-1)}}$

d) $x''(t) + 2x(t) = e^{-t}\cdot u(t-1)$ $x(0+)=0$, $x'(0+)=0$

Vi laplacetransformerer ligningen. La $X(s) = \mathcal{L}[x(t)]$

$$\left[s^2X(s) - s\,x(0+) - x'(0+)\right] + 2X(s) = e^{-1}\frac{1}{s+1}\,e^{-s}$$

idet $\mathcal{L}\left[e^{-t}u(t-1)\right] = e^{-s}\cdot\mathcal{L}\left[e^{-(t+1)}u(t)\right] = e^{-s}\cdot e^{-1}\cdot\frac{1}{s+1}$

Da blir lign. slik

$$\left[s^2 X(s) - 1\right] + 2X(s) = e^{-1}\frac{1}{s+1}e^{-s}$$

$$(s^2+2)\,X(s) = e^{-1}\frac{1}{s+1}e^{-s} + 1$$

$$\boxed{X(s) = e^{-1}\frac{1}{s^2+1}\cdot\frac{1}{s+1}e^{-s} + \frac{1}{s^2+2}}$$

Men fra c) (iv) $e^{-1}\cdot\frac{1}{s^2+1}\cdot\frac{1}{s+1}e^{-s} \Leftrightarrow e^{-1}\cdot g(t-1)$

og av tabell $\frac{1}{s^2+2} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+2} \Leftrightarrow \frac{1}{\sqrt{2}}\sin\sqrt{2}t\cdot u(t)$.

Altså løsning:

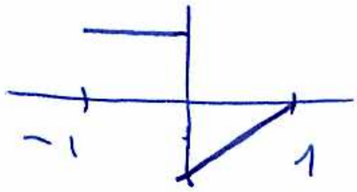
$$\underline{\underline{x(t) = \frac{1}{e}\cdot g(t-1) + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\cdot u(t)}}$$

med $\underline{\underline{g(t) = \left[-\frac{1}{3}\cos\sqrt{2}t + \frac{1}{3\sqrt{2}}\sin\sqrt{2}t + \frac{1}{3}e^{-t}\right]u(t)}}$

Nr 4

$$f(t) = \begin{cases} 1 & -1 < t < 0 \\ t-1 & 0 < t < 1 \end{cases} \qquad f(t+2) = f(t)$$

graf f på $\langle -1, 1 \rangle$



a) $f(7/3) = f(\frac{7}{3} - 2) = f(\frac{1}{3}) = \frac{1}{3} - 1 = \underline{\underline{-\frac{2}{3}}}$

utnytter at f er 2-periodisk

$f(102) = f(102 - 2 \cdot 51) = \underline{\underline{f(0)}}$ $\quad$ $\underline{\underline{f(0) \text{ er uspesifisert}}}$

Går vi 51 perioder tilbake fra 102, havner vi i 0.

La $s(t)$ = summen av fourier-rekken for t.

Da: $s(\frac{7}{3}) = f(7/3) = \underline{\underline{-\frac{2}{3}}}$ $\quad$ $\underline{\underline{\text{fordi } f \text{ er kont. i } \tfrac{1}{3}}}$

og $s(102) = s(0) = \dfrac{f(0+) + f(0-)}{2} = \dfrac{1}{2}\left[-1 + 1\right] = \underline{\underline{0}}$

b) $$f(t) = a(0) + \sum_{n=1}^{\infty}\left[a(n)\cos n\omega t + b(n)\sin n\omega t\right]$$

$\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{2} = \pi$ $\quad$ idet $T = 2$ (perioden)

$$a(0) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{1}{2}\int_{-1}^{1} f(t)\,dt = \frac{1}{2}\cdot\underbrace{\left\{1\cdot 1 - \frac{1}{2}1\cdot 1\right\}}_{\text{areal}}$$

$$a(0) = \frac{1}{2}\left[1 - \frac{1}{2}\right] = \underline{\frac{1}{4}}$$

$$\begin{cases} a(n) = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt = \frac{2}{2}\int_{-1}^{1} f(t)\cos n\pi t\,dt \\ \frac{T}{2} = \frac{2}{2} = 1 \qquad \omega = \pi \end{cases}$$

$$a(n) = \int_{-1}^{0} 1\cdot\cos n\pi t\,dt + \int_{0}^{1} \underset{u\cdot}{(t-1)}\underset{v'}{\cos n\pi t}\,dt =$$

$$= \left[\frac{\sin n\pi t}{n\pi}\right]_{-1}^{0} + \left[(t-1)\cdot\frac{\sin n\pi t}{n\pi}\right]_{0}^{1} - \int_{0}^{1} 1\cdot\frac{\sin n\pi t}{n\pi}\,dt =$$

$$= 0 + 0 - \frac{1}{n\pi}\left[\frac{-\cos n\pi t}{n\pi}\right]_{0}^{1} = \frac{1}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

Så finner vi $b(n)$ og setter $\sin n\pi t$ istedetfor $\cos n\pi t$

$$b(n) = \int_{-1}^{0} 1\cdot\sin n\pi t\,dt + \int_{0}^{1} (t-1)\sin n\pi t\,dt =$$

$$= \left[\frac{-\cos n\pi t}{n\pi}\right]_{-1}^{0} + \left[(t-1)\cdot\frac{-\cos n\pi t}{n\pi}\right]_{0}^{1} - \int_{0}^{1} 1\cdot\frac{-\cos n\pi t}{n\pi}\,dt$$

$$b(n) = \frac{-1}{n\pi} + \frac{\cos(-n\pi)}{n\pi} + \left[0-(0-1)\cdot\frac{\cos 0}{n\pi}\right] + \frac{1}{n\pi}\left[\frac{\sin n\pi t}{n\pi}\right]_0^1$$

$$= -\frac{1}{n\pi} + \frac{\cos n\pi}{n\pi} - \frac{1}{n\pi} + \frac{1}{n^2\pi^2}[0-0]$$

$$= \frac{\cos n\pi - 2}{n\pi}.$$

Fourier-rekken:

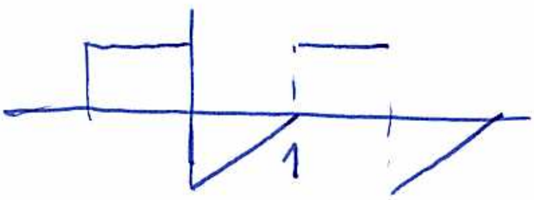
$$\frac{1}{4} + \sum_{n=1}^{\infty}\left\{\frac{\cos n\pi - 1}{n^2\pi^2}\cos n\pi t + \frac{\cos n\pi - 2}{n\pi}\sin n\pi t\right\}$$

c) $\underline{t=1}$ Da får vi

$$\frac{1}{4} + \sum_{n=1}^{\infty}\left\{\frac{\cos n\pi - 1}{n^2\pi^2}\cos n\pi + \frac{\cos n\pi - 2}{n\pi}\underbrace{\sin n\pi}_{=0}\right\}$$

$$= \frac{1}{4} + \sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2\pi^2} \qquad \left((\cos n\pi)^2 = 1\right)$$

Vi vet at $s(1) = \frac{f(1+)+f(1-)}{2} = \frac{1+0}{2} = \frac{1}{2}$.



Altså
$$\boxed{\frac{1}{2} = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1-\cos n\pi}{n^2\pi^2}}$$

$\cos n\pi = (-1)^n$ ⇒ $1-\cos n\pi = \begin{cases} 1-1 & n \text{ partall} \\ 1-(-1) & n \text{ odde} \end{cases}$

Vi skriver ut leddene i rekken og får da:

$$\frac{1}{2} = \frac{1}{4} + \Big[\underset{n=1}{\frac{2}{\pi^2}} + \underset{n=2}{0} + \underset{n=3}{\frac{2}{3^2\pi^2}} + \underset{n=4}{0} + \underset{n=5}{\frac{2}{5^2\pi^2}} + \cdots$$

$$\frac{1}{4} = \frac{2}{\pi^2}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] \Big| \cdot \frac{\pi^2}{2}$$

⇒ $$\underline{\underline{\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots}}$$